Methodology for the Construction and Enhancement of Risk-Parity Portfolios

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Abstract—The investment capital of a Risk-Parity (RP) portfolio is allocated in such a way that all portfolio constituents should contribute equally to the total “risk” of the portfolio. This is in contrast to more conventional asset management, such as equally weighted funds, in which capital (rather than risk) is distributed equally. While the RP concept is straightforward, the implementation is less so, and is described here in detail.

The economic rationale behind Risk-Parity is to enforce diversification, with the goal of increasing risk adjusted return on capital invested. Our RP construction, when applied to a managed futures portfolio, demonstrates improvement upon equal weighted allocation and other suitable benchmarks. Performance can be significantly enhanced by the overlay of market views on the asset allocator, a key innovation over conventional RP-Funds, which we name Active Risk-Parity (ARP).

I. INTRODUCTION

The interest of the signal processing community in utilizing their methods for investment decision making and risk management has lagged behind the enthusiasm of other engineering disciplines (e.g. control theory or artificial intelligence) in migrating to applications of quantitative finance.

However, a recent surge in publication activity, culminating with a special issue of IEEE Signal Processing Magazine [1], has changed the status quo. Financial Engineering has become a mainstream subject for signal processing practitioners. With this in mind, the current work addresses an important concept in portfolio management, namely how to enforce a systematic diversity on investments to protect from concentrated risk.

For a portfolio of \( N \) financial assets with fractions (weights) \( w_k(t) \) of the total capital allocated to asset \( k \) at time \( t \) for \( k = 1, 2, \ldots, N \), we define its risk as the dynamic standard deviation of portfolio returns, also known as volatility:

\[
\sigma_P(t) = \sqrt{w^T(t) \Sigma_P(t) w(t)},
\]

where the chosen vector of portfolio weights is

\[
w(t) = [w_1(t), w_2(t), \ldots, w_N(t)]^T
\]

and \( \Sigma_P(t) \) is the time-varying, symmetric, positive semidefinite covariance matrix of constituent asset returns. Time dependence is dropped from further notation, it being understood that \( \Sigma_P \) and \( w \) are re-estimated and recalculated, respectively, as required for re-balancing.

In order to achieve risk parity, the weights \( w_k \) must be chosen such that the contribution of each asset to total risk \( \sigma_P \) is equal [2]. This requirement can be approximated by the constraint

\[
\frac{\partial \sigma_P^2}{\partial w_j} = \frac{\partial \sigma_P^2}{\partial w_k} \quad \forall j, k \in \{1, 2, \ldots, N\}
\]

as shown later. There are also constraints on the weights \( w_k \) which can be chosen to meet this requirement. For example:

- long only: \( w_k \geq 0 \quad \forall k \)
- no leverage: \( \sum_k |w_k| \leq 1 \)
- capped leverage: \( \sum_k |w_k| \leq L \)
- position limits: \( P_k^l \leq w_k \leq P_k^u \)
- risk budget: \( w^T \Sigma_P w \leq \sigma_B^2 \)

This analysis examines a long-only managed futures portfolio, subject to a leverage cap \( L \) and risk budget \( \sigma_B \). To allow comparison of the RP-allocator performance with its benchmarks, the mean annualized return, annualized volatility and Sharpe Ratio are reported as quant metrics.

Section II presents the optimization required to achieve risk parity. The enhancement of the strategy with market views is explained in Section III. The performance of the RP-algorithm is examined in detail in Section IV. The paper concludes with an analysis of the strengths and weaknesses of RP-Funds, and possible research directions for their improvement.

II. RISK PARITY METHODOLOGY

A. Risk Attribution

The marginal-risk of a portfolio constituent is defined as the change in the portfolio volatility \( \sigma_P \) caused by an infinitesimal change in the weight \( w_k \) allocated to it:

\[
R_{k}^{\text{marginal}} \overset{\text{def}}{=} \frac{\partial \sigma_P}{\partial w_k} = \frac{1}{2 \sigma_P} \frac{\partial \sigma_P^2}{\partial w_k}
\]

The chain rule is used above to express marginal risk as a partial derivative of variance rather than volatility, as later this avoids square roots and simplifies the optimization. The first major assumption made in the risk parity framework is that the total risk contributed by position \( k \), often known as component-risk, is given by [3]

\[
R_{k}^{\text{component}} \overset{\text{def}}{=} w_k R_{k}^{\text{marginal}} = \frac{w_k}{2 \sigma_P} \frac{\partial \sigma_P^2}{\partial w_k}
\]
By defining the selection vector $e_k \overset{\text{def}}{=} [0, \ldots, 0, 1, \ldots, 0]^T$ which is all zeros except for a unit value at the $k$th position, we get:

$$\sum_k R_k^\text{COMPONENT} = \sum_k w_k R_k^\text{MARGINAL}$$

$$= \frac{1}{2\sigma_p} \sum_k w_k \frac{\partial w_k^T \Sigma_p w}{\partial w_k}$$

$$= \frac{1}{2\sigma_p} \sum_k 2w_k^T \Sigma_p e_k w_k$$

$$= \frac{2w^T \Sigma_p w}{2\sigma_p}$$

$$= \sigma_p.$$  (10)

In words, the portfolio risk is the sum of its component risks. In fact this is true for any homogenous risk measure. For the presented RP-framework is relatively straight-forward to formulate for an optimizer, for example in MATLAB. The remaining challenge is to ensure that the required covariance matrix of asset returns $\Sigma_p$, is estimated in a robust and reliable manner. The topic of covariance matrix estimation is not covered here.

### B. Portfolio Construction

Having the objective of minimizing the “deviation from risk parity” under allocation constraints, we can frame a cost function for the optimization problem as the sum of squared component-risk differences:

$$J(w) = \sum_{j=1}^N \sum_{k>j} \left( w_k \frac{\partial \sigma^2_p}{\partial w_j} - w_j \frac{\partial \sigma^2_p}{\partial w_k} \right)^2.$$  (12)

From eqn. (8) this becomes

$$J(w) = \sum_{j=1}^N \sum_{k>j} \left( w_k^T \Sigma_p w_j \right)^2,$$  (13)

up to a scale factor, where $E_{j,k} \overset{\text{def}}{=} \text{diag}(e_j) - \text{diag}(e_k)$. The optimization problem is to find the RP-portfolio $w_{RP}$ which minimizes $J(w)$ subject to appropriate constraints, which we take to be:

$$w_{RP} = \arg\min_w \left\{ \sum_{j=1}^N \sum_{k>j} \left( w_k^T \Sigma_p E_{j,k} w \right)^2 \right\}$$

s.t. $w_k \geq 0 \ \forall k$ and $\sum_k w_k = 1$  (14)

In order to facilitate convergence of the solvers used for portfolio allocation, the gradient and Hessian of the cost function with respect to the weights can be computed up to a scale factor as:

$$\nabla J(w) = \sum_{j} \sum_{k>j} \left( w_k^T \Sigma_p E_{j,k} w \right) \left( \Sigma_p E_{j,k} + E_{j,k} \Sigma_p \right) w$$

$$H_J(w) = \sum_{j} \sum_{k>j} \left( w_k^T \Sigma_p E_{j,k} w \right) \left( \Sigma_p E_{j,k} + E_{j,k} \Sigma_p \right) + \left( \Sigma_p E_{j,k} + E_{j,k} \Sigma_p \right) w w^T \left( \Sigma_p E_{j,k} + E_{j,k} \Sigma_p \right).$$  (15)

The presented RP-framework is relatively straight-forward to formulate for an optimizer, for example in MATLAB. The remaining challenge is to ensure that the required covariance matrix of asset returns $\Sigma_p$, is estimated in a robust and reliable manner. The topic of covariance matrix estimation is not covered here.

![Risk Parity Cost Function for Two Asset Portfolio](image)

Fig. 1. Logarithmic Contour plot of the RP Squared-Error Surface. For no leverage portfolio, Risk-Parity is closest to being achieved at the four portfolios where the X-shaped trough meets the unit circle.

### C. Theoretical Considerations

The cost function $J(w)$, which quadratically penalizes deviation from risk parity, does not have a unique global minimum. Fig. 1 shows the squared-error surface for a synthetic two asset portfolio whose constituents have normally distributed returns correlated at around 50%. A no-leverage constraint forces the solution set to lie on the unit-circle shown. Volatility $\sigma_p$ is a two-sided risk measure, which does not distinguish between upside and downside risk. As a result the long-short RP-portfolio seems to have an equivalent short-long allocation with the same risk performance. In fact there are four valid RP-portfolios. These allocations do not deliver similar returns: the direction of market moves matter, which is why it is recommended to employ Active-RP (Section III).

Note from eqn. (5) that component risk can be negative which corresponds to a position which acts as a hedge to
the portfolio. Such hedging positions are excluded from RP-portfolios, since by definition, all component risks must be equal and therefore have the same sign.

The application of leverage and portfolio-risk constraints has been omitted from the optimization of eqn. (14) because they can be applied post-hoc as a result of both capital and risk being linear in the portfolio weights.

III. Active Risk Parity

The Passive Risk Parity approach to fund management ignores your market views, whether bullish or bearish. You will sometimes be invested long in what you may consider to be a falling market. This undesirable situation can be avoided through appropriate utilization of a view overlay, a method we term Active Risk-Parity (ARP). This is implemented as follows:

1) Ensure your investable universe is large to enable portfolio diversification
2) At re-balancing, narrow your investable subset to only those assets on which you currently have strong views
3) Retain Risk-Parity with volatility control as the portfolio construction mechanism for those instruments which are invested: eqn. (14).

ARP has two main advantages over passive-RP. The first is, clearly, that the value of “good” traders or view-generators is not discarded. The second advantage is more subtle. The problem with large portfolios is that if every asset must receive an allocation, then performance will be very similar to that of the equal-allocation portfolio. For this reason, the benefits of RP asset management are most noticeable on smaller portfolios, which have greater scope to deviate from equi-weighted funds (EWFs). ARP portfolios are constructed from a smaller investable universe by design, and thus have more degrees of freedom to out-perform PRP when benchmarked against EWFs.

In the back-tests conducted for this study, a threshold based exponentially weighted mean-reverting metric on daily returns for commodity markets was used for our view generator. This reflects observed mean-reverting behavior in many commodity markets, with the assumption that prices should revert to the marginal cost of production. Any other view generator can be incorporated in ARP. However, view-generation and market forecasting in general are beyond the scope of the current report.

IV. Back-Testing

A. Data

For the purposes of back-testing the RP-allocator performance on realized market-data, an investable universe of 26 commodity futures was chosen from 5 different sectors, detailed in Table I. In addition, the RP-performance was benchmarked against:

1) an equal-weighted portfolio selected from the same universe as the RP-portfolio
2) the S&P 500, a capitalization-weighted index based on the stock prices of 500 American companies
3) the DJUBS Commodity Index, a broadly diversified index on 19 commodity futures in seven sectors

B. Parameters

The RP-portfolio base currency is USD. Market-data is taken as the previous trading-day close prices on the front-month futures for each commodity. Daily returns are used for the computation of the covariance matrix $\Sigma_p$ at re-balancing, which occurs on the last trading day of each month. The back-test is conducted over the 12-years preceding June 2012. The risk budget is an annualized volatility of 15%, and leverage is capped at 150% of capital.

In order that the performance results incorporate realistic rolling and re-balancing costs, there is a conservative charge of 80-basis-points on all transactions, which includes the fees for selling all expiring contracts and taking longer-dated positions.

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**Table I: Universe of investable assets for RP back-testing**

<table>
<thead>
<tr>
<th>Sector</th>
<th>Commodity</th>
<th>Exchange</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grains</td>
<td>Wheat</td>
<td>CBOT</td>
</tr>
<tr>
<td>Grains</td>
<td>Corn</td>
<td>CBOT</td>
</tr>
<tr>
<td>Grains</td>
<td>Soybeans</td>
<td>CBOT</td>
</tr>
<tr>
<td>Grains</td>
<td>Soybean Oil</td>
<td>CBOT</td>
</tr>
<tr>
<td>Grains</td>
<td>Soybean Meal</td>
<td>CBOT</td>
</tr>
<tr>
<td>Softs</td>
<td>Cocoa</td>
<td>ICE</td>
</tr>
<tr>
<td>Softs</td>
<td>Coffee</td>
<td>ICE</td>
</tr>
<tr>
<td>Softs</td>
<td>Sugar</td>
<td>ICE</td>
</tr>
<tr>
<td>Softs</td>
<td>Orange Juice</td>
<td>ICE</td>
</tr>
<tr>
<td>Softs</td>
<td>Cotton</td>
<td>ICE</td>
</tr>
<tr>
<td>Energy</td>
<td>Gas</td>
<td>ICE</td>
</tr>
<tr>
<td>Energy</td>
<td>Brent Crude</td>
<td>ICE</td>
</tr>
<tr>
<td>Energy</td>
<td>WTI Crude Oil</td>
<td>NYMEX</td>
</tr>
<tr>
<td>Energy</td>
<td>Heating Oil</td>
<td>NYMEX</td>
</tr>
<tr>
<td>Energy</td>
<td>Natural Gas</td>
<td>NYMEX</td>
</tr>
<tr>
<td>Industrial</td>
<td>Aluminium</td>
<td>LME</td>
</tr>
<tr>
<td>Industrial</td>
<td>Lead</td>
<td>LME</td>
</tr>
<tr>
<td>Industrial</td>
<td>Copper</td>
<td>LME</td>
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<tr>
<td>Industrial</td>
<td>Nickel</td>
<td>LME</td>
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<td>Industrial</td>
<td>Zinc</td>
<td>LME</td>
</tr>
<tr>
<td>Precious</td>
<td>Gold</td>
<td>COMEX</td>
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<tr>
<td>Precious</td>
<td>Silver</td>
<td>COMEX</td>
</tr>
<tr>
<td>Precious</td>
<td>Platinum</td>
<td>NYMEX</td>
</tr>
<tr>
<td>Precious</td>
<td>Palladium</td>
<td>NYMEX</td>
</tr>
</tbody>
</table>

**Table II: Quant Metrics of RP and Benchmark Performance**

<table>
<thead>
<tr>
<th>Quant Metrics</th>
<th>PRP</th>
<th>ARP</th>
<th>EWF</th>
<th>S&amp;P</th>
<th>DJUBS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. Return p.a. (%)</td>
<td>6.53</td>
<td>11.47</td>
<td>6.13</td>
<td>1.29</td>
<td>5.24</td>
</tr>
<tr>
<td>Volatility p.a. (%)</td>
<td>16.39</td>
<td>15.84</td>
<td>15.84</td>
<td>21.33</td>
<td>17.91</td>
</tr>
<tr>
<td>Sharpe Ratio (2%)</td>
<td>0.27</td>
<td>0.60</td>
<td>0.26</td>
<td>0.03</td>
<td>0.18</td>
</tr>
</tbody>
</table>
C. Performance

Fig. 2 shows the net asset value as a percentage of the starting capital for each of the back-tested funds and benchmarks. Table II provides a more clear perspective on their comparative merits. The history of the allocations over time for PRP and ARP are shown in Fig. 3. Finally the dynamic volatility charted in Fig. 4.

The Passive RP fund has very similar performance to an equal weighted fund, because, as explained earlier, every asset must receive an allocation and the universe is large. This is illustrated more clearly in Fig. 3. While the PRP fund is clearly not equally weighted, it is not too far away from it, resulting in roughly equivalent risk adjusted returns.

ARP achieves double-digit annualized returns over the interval, alone in its peer group. Part of this performance can be attributed to the view generator, but risk control is obviously working with a Sharpe-Ratio twice its nearest competitor. It is clear from the lower part of Fig. 3 that the ARP weights have a greater domain over which to optimize than in PRP.

Where the allocator clearly achieves superior results, is in maintaining a more stable portfolio volatility, despite the challenge of estimating dynamic volatility and covariance.

V. DISCUSSION AND CONCLUSION

Our main conclusion is that the presented methodology for portfolio risk management is relatively convenient to implement, and displays improved risk-return characteristics over more conventional allocations in our back-tests.

If the returns on all assets available for portfolio formation are jointly elliptically distributed, then all portfolios can be characterized completely by their mean and standard deviation. The derived portfolio optimization will achieve Risk-Parity under these general conditions. An interesting line of further research would be the construction of a Value-at-Risk (VaR)-Parity portfolio of assets with asymmetric returns, since the methodology has only been derived for the two-sided risk measure of volatility.

The RP error surface shows a number of disadvantages. Firstly, the multiple long-short allocation possibilities are clearly a danger. It seems that without views, passive Risk-Parity should be applied to a long-only portfolio, if it is used at all. In addition to having non-global optimum, the error-surface also has shallow gradient around the minima (contours are logarithmic in Fig. 1). This disadvantage, which slows the convergence of the RP-allocator, results from a cost function which is quartic in the coefficients. A less extreme cost function may be worth researching.

REFERENCES